It's About Time: Teaching Correct Intuition For General Relativity

American Association of Physics Teachers: 2018 Winter Meeting

Jonathan Matthew Clark

The University of Tennessee, Knoxville

jclar121@vols.utk.edu

TN Tennessee Higher Education Commission

Presentation Outline

Common analogies

Some common analogies used when teaching general relativity include:

Common analogies

Some common analogies used when teaching general relativity include:

- "Gravity comes from warps in space."

Common analogies

Some common analogies used when teaching general relativity include:

- "Gravity comes from warps in space."
- "The earth curves space like a bowling ball on a trampoline."

Common analogies

Some common analogies used when teaching general relativity include:

- "Gravity comes from warps in space."
- "The earth curves space like a bowling ball on a trampoline."

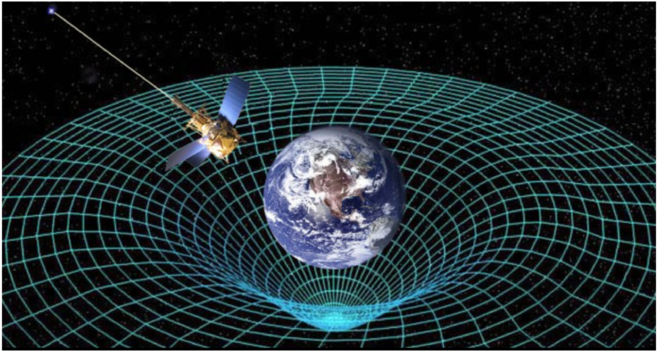
While sometimes useful, they can carry incorrect connotations.

Common analogies

Some common analogies used when teaching general relativity include:

- "Gravity comes from warps in space."
- "The earth curves space like a bowling ball on a trampoline."

While sometimes useful, they can carry incorrect connotations.



Conceptual framework

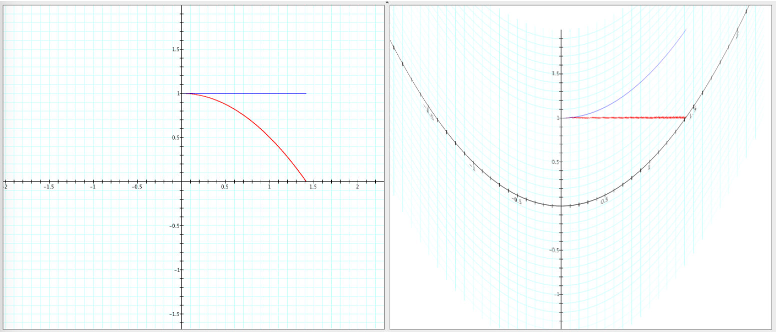
- Albert Einstein realized that free-fall is inertial motion.

Conceptual framework

- Albert Einstein realized that free-fall is inertial motion.
- This means gravitational forces are the result of changing coordinates.

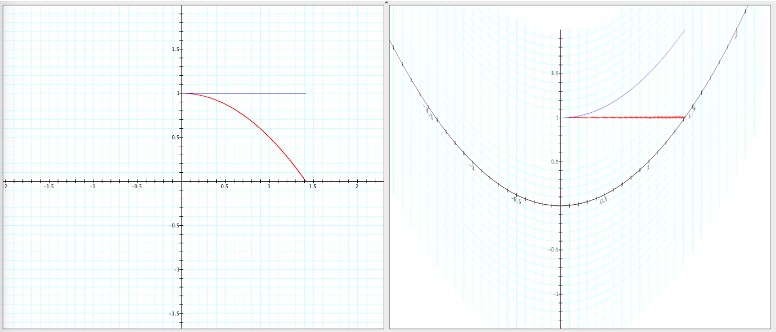
Conceptual framework

- Albert Einstein realized that free-fall is inertial motion.
- This means gravitational forces are the result of changing coordinates.



Conceptual framework

- Albert Einstein realized that free-fall is inertial motion.
- This means gravitational forces are the result of changing coordinates.



An accurate analogy for gravitation is the fake centrifugal force outward you feel when you jump on a fast roundabout, or when you turn your steering wheel hard and bump into your car window.

Presentation Outline

Principle of correspondence

- Students might wonder how general relativity models simple situations that they are familiar with.

Principle of correspondence

- Students might wonder how general relativity models simple situations that they are familiar with.
- The simplest non-trivial situation we could consider is our life of slow velocities and weak gravitational fields on earth.

Principle of correspondence

- Students might wonder how general relativity models simple situations that they are familiar with.
- The simplest non-trivial situation we could consider is our life of slow velocities and weak gravitational fields on earth.
- General relativity better be able to handle this situation, else it will fail the principle of correspondence.

Weak-field limit

The following is a reformulation of an important result called the weak-field limit of general relativity, which is proven in the appendix of this presentation:

Weak-field limit

The following is a reformulation of an important result called the weak-field limit of general relativity, which is proven in the appendix of this presentation:

Theorem

Suppose the Minkowski metric's time-component is perturbed by a radially symmetric, time-independent function $\phi(r)$ such that the geodesic equations reproduce Newton's universal law of gravitation. Then

$$\phi(r) = -\frac{2GM}{c^2 r}$$

and so the metric's time-component agrees with the Schwarzschild metric:

$$g_{00}(r) = 1 - \frac{2GM}{c^2 r}.$$

Weak-field limit

This tells us that nearly all of gravitational acceleration we experience on earth is due to the curvature of time.

- "Gravity comes from warps in time and space."

Weak-field limit

This tells us that nearly all of gravitational acceleration we experience on earth is due to the curvature of time.

- "Gravity comes from warps in time and space."

Spatial curvature contributes more to the exotic effects of relativity.

Weak-field limit

This tells us that nearly all of gravitational acceleration we experience on earth is due to the curvature of time.

- "Gravity comes from warps in time and space."

Spatial curvature contributes more to the exotic effects of relativity.

Presentation Outline

Extrinsic or Intrinsic curvature?

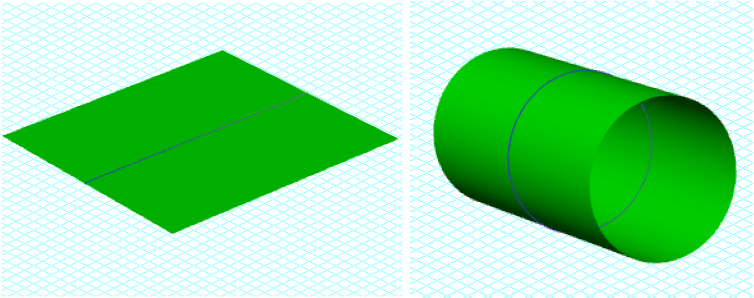
- Extrinsic curvature has to do with the way surfaces are embedded into larger spaces. Their geodesics may be just straight lines.

Extrinsic or Intrinsic curvature?

- Extrinsic curvature has to do with the way surfaces are embedded into larger spaces. Their geodesics may be just straight lines.
- A cylinder is not intrinsically curved because it can be smoothly flattened.

Extrinsic or Intrinsic curvature?

- Extrinsic curvature has to do with the way surfaces are embedded into larger spaces. Their geodesics may be just straight lines.
- A cylinder is not intrinsically curved because it can be smoothly flattened.



Extrinsic or Intrinsic curvature?

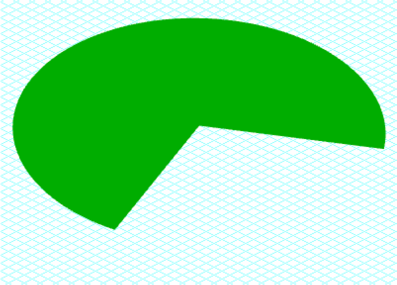
- Intrinsic curvature has to do with the intrinsic properties of the surface, like how distances and angles are measured.

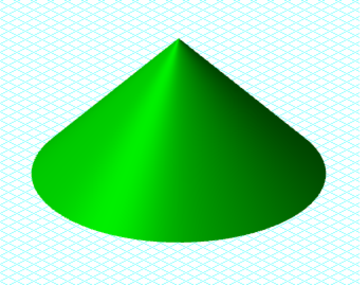
Extrinsic or Intrinsic curvature?

- Intrinsic curvature has to do with the intrinsic properties of the surface, like how distances and angles are measured.
- A cone is only curved intrinsically at the apex, which can be found by drawing geodesic triangles on the cone's surface and examining the sum of their interior angles.

Extrinsic or Intrinsic curvature?

- Intrinsic curvature has to do with the intrinsic properties of the surface, like how distances and angles are measured.
- A cone is only curved intrinsically at the apex, which can be found by drawing geodesic triangles on the cone's surface and examining the sum of their interior angles.





Einstein's field equations

- The vacuum in general relativity satisfies the deceptively simple Einstein field equations:

Einstein's field equations

- The vacuum in general relativity satisfies the deceptively simple Einstein field equations: $R_{uv} = 0$.

Einstein's field equations

- The vacuum in general relativity satisfies the deceptively simple Einstein field equations: $R_{uv} = 0$.
- There's a deep reason why relativity couldn't model gravitation without time and all three spatial dimensions being intertwined.

Einstein's field equations

- The vacuum in general relativity satisfies the deceptively simple Einstein field equations: $R_{uv} = 0$.
- There's a deep reason why relativity couldn't model gravitation without time and all three spatial dimensions being intertwined.
- Differential geometry is highly sensitive to the number of dimensions.

Einstein's field equations

- The vacuum in general relativity satisfies the deceptively simple Einstein field equations: $R_{uv} = 0$.
- There's a deep reason why relativity couldn't model gravitation without time and all three spatial dimensions being intertwined.
- Differential geometry is highly sensitive to the number of dimensions.

Theorem

Suppose a spacetime manifold $\mathcal{M}$ has strictly less than four dimensions. If $R_{uv} = 0$ everywhere, then $\mathcal{M}$ has zero total curvature.

Presentation Outline

Bibliography

- Ferraro, R. (2007). *Einstein's Space-Time: An Introduction to Special and General Relativity*. Springer.
- Misner C. W., Thorne, K. S., & Wheeler J. A. (1973). *Gravitation*. W. H. Freeman and Company.
- [World Science Festival]. (2015, November 25). *Brian Greene Explores General Relativity in His Living Room*. Retrieved from https://www.youtube.com/watch?v=uRijc-AN-F0
- [Science Magazine]. (2015, March 6). *General Relativity: A super-quick, super-painless guide to the theory that conquered the universe.* Retrieved from: http://spark.sciencemag.org/generalrelativity/

Presentation Outline

Appendix: proof of 1st theorem

Proof

We assume that the metric tensor takes the form:

Appendix: proof of 1st theorem

Proof

We assume that the metric tensor takes the form:

$$\mathbf{g} = \begin{pmatrix} 1+\phi(r) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Appendix: proof of 1st theorem

Proof

We assume that the metric tensor takes the form:

$$\mathbf{g} = \begin{pmatrix} 1+\phi(r) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The spacetime interval and the ordinary time will be related by:

Appendix: proof of 1st theorem

Proof

We assume that the metric tensor takes the form:

$$\mathbf{g} = \begin{pmatrix} 1+\phi(r) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The spacetime interval and the ordinary time will be related by:

$$d\tau^2 = (1+\phi(r))dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right).$$

Appendix: proof of 1st theorem

Proof

We assume that the metric tensor takes the form:

$$\mathbf{g} = \begin{pmatrix} 1+\phi(r) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The spacetime interval and the ordinary time will be related by:

$$d\tau^2 = (1+\phi(r))dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right).$$

If the velocity is small relative to the speed of light, we also have:

Appendix: proof of 1st theorem

Proof

We assume that the metric tensor takes the form:

$$\mathbf{g} = \begin{pmatrix} 1+\phi(r) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The spacetime interval and the ordinary time will be related by:

$$d\tau^2 = (1+\phi(r))dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right).$$

If the velocity is small relative to the speed of light, we also have:

$$\frac{d\tau^2}{dt^2} = 1 + \phi(r) - \frac{v^2}{c^2} \approx 1 + \phi(r).$$

Appendix: proof of 1st theorem

Proof

If the function $\phi(r)$ is a small perturbation, then we have $d\tau^2 \approx dt^2$. This means that we can replace the spacetime interval with the ordinary coordinate time. The geodesic has components $q^w(t)$ which satisfy the geodesic equation:

Appendix: proof of 1st theorem

Proof

If the function $\phi(r)$ is a small perturbation, then we have $d\tau^2 \approx dt^2$. This means that we can replace the spacetime interval with the ordinary coordinate time. The geodesic has components $q^w(t)$ which satisfy the geodesic equation:

$$\frac{d^2 q^w(t)}{dt^2} = -\Gamma^w_{uv}(\mathbf{x}) \frac{dq^u(t)}{dt} \frac{dq^v(t)}{dt}$$

$$= \frac{1}{2} g^{wz}(\mathbf{x}) \left\{ \partial_z g_{uv}(\mathbf{x}) - \partial_u g_{zv}(\mathbf{x}) - \partial_v g_{zu}(\mathbf{x}) \right\} \frac{dq^u(t)}{dt} \frac{dq^v(t)}{dt}$$

$$= \frac{1}{2} g^{wz}(\mathbf{x}) \partial_z g_{uv}(\mathbf{x}) \frac{dq^u(t)}{dt} \frac{dq^v(t)}{dt} - g^{wz}(\mathbf{x}) \partial_u g_{zv}(\mathbf{x}) \frac{dq^u(t)}{dt} \frac{dq^v(t)}{dt}.$$

Appendix: proof of 1st theorem

Proof

Also note that, since the metric is time-independent and constant except in a single component, this can be greatly simplified:

Appendix: proof of 1st theorem

Proof

Also note that, since the metric is time-independent and constant except in a single component, this can be greatly simplified:

$$\frac{d^2 q^w(t)}{dt^2} = \frac{1}{2} g^{wz}(\mathbf{x}) \partial_z g_{00}(\mathbf{x}) \frac{dq^0(t)}{dt} \frac{dq^0(t)}{dt} - g^{w0}(\mathbf{x}) \partial_u g_{00}(\mathbf{x}) \frac{dq^u(t)}{dt} \frac{dq^0(t)}{dt}$$
$$= \frac{1}{2} g^{wz}(\mathbf{x}) \partial_z \phi(r) \frac{d(ct)}{dt} \frac{d(ct)}{dt} - g^{w0}(\mathbf{x}) \partial_u \phi(r) \frac{dq^u(t)}{dt} \frac{d(ct)}{dt}$$
$$= \frac{c^2}{2} g^{wz}(\mathbf{x}) \partial_z \phi(r) - c g^{w0}(\mathbf{x}) \partial_u \phi(r) \frac{dq^u(t)}{dt} = \frac{c^2}{2} g^{wz}(\mathbf{x}) \partial_z \phi(r) - c g^{w0}(\mathbf{x}) \frac{d\phi(r)}{dt}$$
$$= \frac{c^2}{2} g^{wz}(\mathbf{x}) \partial_z \phi(r) - c g^{w0}(\mathbf{x}) \frac{d\phi(r)}{dr} \frac{dr}{dt} = \frac{c^2}{2} g^{wz}(\mathbf{x}) \partial_z \phi(r) - c g^{w0}(\mathbf{x}) \frac{d\phi(r)}{dr} v.$$

Appendix: proof of 1st theorem

Proof

For a fixed $w \neq 0$, since $g^{ww}(\mathbf{x}) = -1$ and $g^{w0}(\mathbf{x}) = 0$, we obtain:

Appendix: proof of 1st theorem

Proof

For a fixed $w \neq 0$, since $g^{ww}(\mathbf{x}) = -1$ and $g^{w0}(\mathbf{x}) = 0$, we obtain:

$$\frac{d^2 q^w(t)}{dt^2} = \frac{c^2}{2} g^{wz}(\mathbf{x}) \partial_z \phi(r) - c g^{w0}(\mathbf{x}) \frac{d\phi(r)}{dr} v = -\frac{c^2}{2} \partial_w \phi(r).$$

Appendix: proof of 1st theorem

Proof

For a fixed $w \neq 0$, since $g^{ww}(\mathbf{x}) = -1$ and $g^{w0}(\mathbf{x}) = 0$, we obtain:

$$\frac{d^2 q^w(t)}{dt^2} = \frac{c^2}{2} g^{wz}(\mathbf{x}) \partial_z \phi(r) - c g^{w0}(\mathbf{x}) \frac{d\phi(r)}{dr} v = -\frac{c^2}{2} \partial_w \phi(r).$$

So we can view the spatial-components of the geodesic path as a time-dependent Euclidean vector $\mathbf{q}(t)$ which satisfies:

Appendix: proof of 1st theorem

Proof

For a fixed $w \neq 0$, since $g^{ww}(\mathbf{x}) = -1$ and $g^{w0}(\mathbf{x}) = 0$, we obtain:

$$\frac{d^2 q^w(t)}{dt^2} = \frac{c^2}{2} g^{wz}(\mathbf{x}) \partial_z \phi(r) - c g^{w0}(\mathbf{x}) \frac{d\phi(r)}{dr} v = -\frac{c^2}{2} \partial_w \phi(r).$$

So we can view the spatial-components of the geodesic path as a time-dependent Euclidean vector $\mathbf{q}(t)$ which satisfies:

$$\mathbf{a}(t) = \frac{d^2 \mathbf{q}(t)}{dt^2} = -\frac{c^2}{2} \vec{\nabla} \phi(r).$$

Appendix: proof of 1st theorem

Proof

We compare this to Newton's universal law of gravitation:

Appendix: proof of 1st theorem

Proof

We compare this to Newton's universal law of gravitation:

$$\mathbf{a} = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

Appendix: proof of 1st theorem

Proof

We compare this to Newton's universal law of gravitation:

$$\mathbf{a} = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

For the two accelerations to agree, we must have:

Appendix: proof of 1st theorem

Proof

We compare this to Newton's universal law of gravitation:

$$\mathbf{a} = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

For the two accelerations to agree, we must have:

$$-\frac{c^2}{2}\vec{\nabla}\phi(r) = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

Appendix: proof of 1st theorem

Proof

We compare this to Newton's universal law of gravitation:

$$\mathbf{a} = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

For the two accelerations to agree, we must have:

$$-\frac{c^2}{2}\vec{\nabla}\phi(r) = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

This yields a partial differential equation for the perturbation function:

Appendix: proof of 1st theorem

Proof

We compare this to Newton's universal law of gravitation:

$$\mathbf{a} = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

For the two accelerations to agree, we must have:

$$-\frac{c^2}{2}\vec{\nabla}\phi(r) = -\frac{GM}{r^2}\hat{\mathbf{r}}.$$

This yields a partial differential equation for the perturbation function:

$$\vec{\nabla}\phi(r) = \frac{2GM}{c^2 r^2}\hat{\mathbf{r}}.$$

Appendix: proof of 1st theorem

Proof

This can be solved by applying an indefinite line integral:

Appendix: proof of 1st theorem

Proof

This can be solved by applying an indefinite line integral:

$$\phi(r) = \frac{2GM}{c^2} \int \frac{\hat{\mathbf{r}} \cdot d\mathbf{r}}{r^2} = \frac{2GM}{c^2} \int \frac{dr}{r^2} = -\frac{2GM}{c^2 r} + \alpha.$$

Appendix: proof of 1st theorem

Proof

This can be solved by applying an indefinite line integral:

$$\phi(r) = \frac{2GM}{c^2} \int \frac{\hat{\mathbf{r}} \cdot d\mathbf{r}}{r^2} = \frac{2GM}{c^2} \int \frac{dr}{r^2} = -\frac{2GM}{c^2 r} + \alpha.$$

To find the integration constant, we recall that the metric should approach the Minkowski metric at spatial infinity:

Appendix: proof of 1st theorem

Proof

This can be solved by applying an indefinite line integral:

$$\phi(r) = \frac{2GM}{c^2} \int \frac{\hat{\mathbf{r}} \cdot d\mathbf{r}}{r^2} = \frac{2GM}{c^2} \int \frac{dr}{r^2} = -\frac{2GM}{c^2 r} + \alpha.$$

To find the integration constant, we recall that the metric should approach the Minkowski metric at spatial infinity:

$$\lim_{r \to \infty} (\phi(r)) = 0.$$

Appendix: proof of 1st theorem

Proof

This can be solved by applying an indefinite line integral:

$$\phi(r) = \frac{2GM}{c^2} \int \frac{\hat{\mathbf{r}} \cdot d\mathbf{r}}{r^2} = \frac{2GM}{c^2} \int \frac{dr}{r^2} = -\frac{2GM}{c^2 r} + \alpha.$$

To find the integration constant, we recall that the metric should approach the Minkowski metric at spatial infinity:

$$\lim_{r \to \infty} (\phi(r)) = 0.$$

This easily determines the integration constant:

Appendix: proof of 1st theorem

Proof

This can be solved by applying an indefinite line integral:

$$\phi(r) = \frac{2GM}{c^2}\int \frac{\hat{\mathbf{r}}\cdot d\mathbf{r}}{r^2} = \frac{2GM}{c^2}\int \frac{dr}{r^2} = -\frac{2GM}{c^2 r} + \alpha.$$

To find the integration constant, we recall that the metric should approach the Minkowski metric at spatial infinity:

$$\lim_{r\to\infty}(\phi(r)) = 0.$$

This easily determines the integration constant:

$$\lim_{r\to\infty}\left(-\frac{2GM}{c^2 r} + \alpha\right) = \lim_{r\to\infty}\left(-\frac{2GM}{c^2 r}\right) + \lim_{r\to\infty}(\alpha) = \alpha = 0.$$

Appendix: proof of 1st theorem

Proof.

So we conclude that the time-component for the metric agrees with the Schwarzschild metric's time-component:

Appendix: proof of 1st theorem

Proof.

So we conclude that the time-component for the metric agrees with the Schwarzschild metric's time-component:

$$\boxed{g_{00}(\mathbf{x}) = 1 - \frac{2GM}{c^2 r}}$$

Appendix: proof of 1st theorem

Proof.

So we conclude that the time-component for the metric agrees with the Schwarzschild metric's time-component:

$$\boxed{g_{00}(\mathbf{x}) = 1 - \frac{2GM}{c^2 r}}$$

We assumed that the velocity was small compared to the speed of light. This implies the geodesic in the $w = 0$ case is trivial as expected:

Appendix: proof of 1st theorem

Proof.

So we conclude that the time-component for the metric agrees with the Schwarzschild metric's time-component:

$$\boxed{g_{00}(\mathbf{x}) = 1 - \frac{2GM}{c^2 r}}$$

We assumed that the velocity was small compared to the speed of light. This implies the geodesic in the $w = 0$ case is trivial as expected:

$$\frac{d^2 q^0(t)}{dt^2} = \frac{c^2}{2} g^{0z}(\mathbf{x}) \partial_z \phi(r) - c g^{00}(\mathbf{x}) \frac{d\phi(r)}{dr} v = -c g^{00}(\mathbf{x}) \frac{d}{dr} \left(-\frac{2GM}{c^2 r} \right) v$$

$$= -c g^{00}(\mathbf{x}) \left(\frac{2GM}{c^2 r^2} \right) v = -c g^{00}(\mathbf{x}) \left(\frac{2GM}{c r^2} \right) \left(\frac{v}{c} \right) \approx 0.$$

□